KMS STATES FOR THE GENERALIZED GAUGE ACTION ON GRAPH ALGEBRAS

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Abstract. Given a positive function on the set of edges of an arbitrary directed graph $E = (E^0, E^1)$, we define a one-parameter group of automorphisms on the C*-algebra of the graph $C^*(E)$, and study the problem of finding KMS states for this action. We prove that there are bijective correspondences between KMS states on $C^*(E)$, a certain class of states on its core, and a certain class of tracial states on $C_0(E^0)$. We also find the ground states for this action and give some examples.

1. Introduction

Given a directed graph $E = (E^0, E^1)$, we can associate to it a C*-algebra $C^*(E)$, and an interesting problem that arises is to find relations between the algebraic properties of the algebra and the combinatorial properties of the graph [**13**]. One such problem is to determine the set of KMS states for a certain action on the algebra.

Graph algebras are a generalization of Cuntz algebras and Cuntz-Krieger algebras. For the Cuntz algebra, there is a very natural action of the circle, the gauge action, which can be extended to an action of the real line. The KMS states for this action are studied in [**12**] and later generalized to a more general action of the line, that can be thought of as a generalized gauge action [**5**]. The same is done for the Cuntz-Krieger algebras [**4**], [**6**].

Recently there were similar results proven for the C*-algebra associated to a finite graph. This is done in [**9**] for an arbitrary finite graph, in [**8**] for a certain class of finite graphs via groupoid C*-algebras and in [**1**] for the Toeplitz C*-algebra of the graph.

Our goal is to generalize these results to the case of an arbitrary graph. First we analyze which conditions the restrictions of a KMS state to the core of $C^*(E)$ and to $C_0(E^0)$ must satisfy. By using a description of the core as an inductive limit, we can build a KMS state on $C^*(E)$ from a tracial state on $C_0(E^0)$ satisfying the conditions found.

In section 2 we review some of the basic definitions and results about graph algebras as well as the description of the core as an inductive limit. In section 3 we establish the results concerning KMS states, followed by a discussion on ground states in section 4. In section 5, some examples are given.

Partially supported by Funpesquisa/UFSC.

2. Graph algebras

DEFINITION 2.1. A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of nonempty sets E^0, E^1 and functions $r, s : E^1 \to E^0$; an element of E^0 is called a *vertex* of the graph, and an element of E^1 is called an *edge*. For an edge e, we say that $r(e)$ is the *range* of e and $s(e)$ is the *source* of e.

DEFINITION 2.2. A vertex v in a graph E is called a *source* if $r^{-1}(v) = \emptyset$, and is said to be *singular* if it is either a source, or $r^{-1}(v)$ is infinite.

DEFINITION 2.3. A *path of length* n in a graph E is a sequence $\mu = \mu_1\mu_2 \ldots \mu_n$ such that $r(\mu_i + 1) = s(\mu_i)$ for all $i = 1, \ldots, n-1$. We write $|\mu| = n$ for the length of μ and regard vertices as paths of length 0. We denote by E^n the set of all paths of length n and $E^* = \cup_{n\geq 0} E^n$. We extend the range and source maps to E^* by defining $s(\mu) = s(\mu_n)$ and $r(\mu) = r(\mu_1)$ if $n \geq 2$ and $s(v) = v = r(v)$ for $n = 0$.

DEFINITION 2.4. Given a graph E, we define the *C*-algebra of* E as the universal C*-algebra $C^*(E)$ generated by mutually orthogonal projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{s_e\}_{e\in E^1}$ with mutually orthogonal ranges such that

(1) $s_e^* s_e = p_{s(e)}$;
(2) $s_e s_e^* \leq p_r(e)$ for every $e \in E^1$;
(3) $p_v = \sum_{e\in r^{-1}(v)} s_e s_e^*$ for every $v \in E^0$ such that $0 < |r^{-1}(v)| < \infty$.

For a path $\mu = \mu_1 \ldots \mu_n$, we denote the composition $s_{\mu_1} \ldots s_{\mu_n}$ by s_μ, and for $v \in E^0$ we define s_v to be the projection p_v.

Propositions 2.5, 2.6, 2.8 and 2.9 below are found in [**13**] (as Corollary 1.15, Proposition 2.1, Proposition 3.2 and Corollary 3.3, respectively) in the context of row-finite graphs, but their proofs hold just the same for general graphs as above.

PROPOSITION 2.5. *For* $\alpha, \beta, \mu, \nu \in E^*$ *we have*

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_{\mu\alpha'} s_\beta^* & \text{if } \alpha = \nu\alpha' \\ s_\mu s_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise} \end{cases}$$

and $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$.

PROPOSITION 2.6. *Let E be a graph. Then there is an action γ of $\mathbb{T}$ on $C^*(E)$, called a* gauge action, *such that $\gamma_z(s_e) = zs_e$ for every $e \in E^1$ and $\gamma_z(p_v) = p_v$ for every $v \in E^0$.*

DEFINITION 2.7. The *core* of the algebra $C^*(E)$ is the fixed-point subalgebra for the gauge action, denoted by $C^*(E)^\gamma$.

PROPOSITION 2.8. $C^*(E)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu),\ |\mu| = |\nu|\}$.

PROPOSITION 2.9. *There is a conditional expectation $\Phi : C^*(E) \to C^*(E)^\gamma$ such that $\Phi(s_\mu s_\nu^*) = [|\mu| = |\nu|] s_\mu s_\nu^*$.*

It is useful to describe the core as an inductive limit of subalgebras, as was done in an appendix in [**2**]. The idea is as follows. For $k \geq 0$ define the sets

$$F_k = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu)\},$$
$$\mathcal{E}_k = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^k \text{ and } s(\mu) = s(\nu) \text{ is singular}\},$$
$$C_k = F_0 + \cdots + F_k.$$

Also, for a given vertex v we define

$$F_k(v) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu) = v\}$$

so that

$$F_k = \bigoplus_{v \in E^0} F_k(v) \tag{2.1}$$

as a direct sum of C*-algebras.

Lemma 2.10. *Let Λ be the set of all finite subsets of E^k and for $\lambda \in \Lambda$ define*

$$u_\lambda = \sum_{\mu \in \lambda} s_\mu s_\mu^*.$$

Then $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit of F_k consisting of projections.

Proof. This is a direct consequence of Proposition 2.5. □

The following result is a combination of Proposition A.1 and Lemma A.2 in [**2**].

Proposition 2.11. *With the notation as above for a graph E, the following hold for $k \geq 0$:*

(a) *C_k is a C*-subalgebra of $C^*(E)^\gamma$, F_{k+1} is an ideal in C_k, $C_k \subseteq C_{k+1}$ and*
$$C^*(E)^\gamma = \varinjlim C_k.$$
(b) $F_k \cap F_{k+1} = \bigoplus\{F_k(v) : 0 < |r^{-1}(v)| < \infty\}$. *(C*-algebraic direct sum)*
(c) $C_k = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1} \oplus F_k$ *(vector space direct sum)*

With the above, we can now prove the following.

Proposition 2.12. *For each $k \geq 0$, $C_k \cap F_{k+1} = F_k \cap F_{k+1}$.*

Proof. Obviously one has $F_k \cap F_{k+1} \subseteq C_k \cap F_{k+1}$. On the other hand, given $x \in C_k \cap F_{k+1}$, one can decompose x as sums in C_k and C_{k+1} with Proposition 2.11(c), use the fact that $F_k = \mathcal{E}_k \oplus F_k \cap F_{k+1}$ and the uniqueness of the direct sum decompositions of x to conclude that $x \in F_k$. □

3. KMS states for the generalized gauge action

In this section we define an action on $C^*(E)$ from a function $c : E^1 \to \mathbb{R}_+^*$ similar to what is done in [**5**] for the Cuntz algebras and in [**6**] for the Cuntz-Krieger algebras. We will always suppose that there is a constant $k > 0$ such that $c(e) > k$ for all $e \in E^1$ and that $\beta > 0$. Observe that in this case $c^{-\beta}$ is bounded.

We extend a function as above to a function $c : E^* \to \mathbb{R}_+^*$ by defining $c(v) = 1$ if $v \in E^0$ and $c(\mu) = c(\mu_1)\ldots c(\mu_n)$ if $\mu = \mu_1 \cdots \mu_n \in E^n$.

Proposition 3.1. *Given a function $c : E^1 \to \mathbb{R}_+^*$, there is a strongly continuous action $\sigma^c : \mathbb{R} \to \mathrm{Aut}(C^*(E))$ given by $\sigma_t^c(p_v) = p_v$ for all $v \in E^0$ and $\sigma_t^c(s_e) = c(e)^{it} s_e$ for all $e \in E^1$.*

Proof. Let $T_e = c(e)^{it} s_e$ and note that T_e is a partial isometry with $T_e^* T_e = s_e^* s_e$ and $T_e T_e^* = s_e s_e^*$. It follows that the sets $\{p_v\}_{v\in E^0}$ and $\{T_e\}_{e\in E^1}$ satisfy the same relations as $\{p_v\}_{v\in E^0}$ and $\{s_e\}_{e\in E^1}$. By the universal property, there is a homomorphism $\sigma_t^c : C^*(E) \to C^*(E)$ such that $\sigma_t^c(p_v) = p_v$ for all $v \in E^0$ and $\sigma_t^c(s_e) = T_e = c(e)^{it} s_e$ for all $e \in E^1$.

It is easy to see that $\sigma^c_{t_1}\circ\sigma^c_{t_2}=\sigma^c_{t_1+t_2}$ and $\sigma^c_0=Id$. Hence σ^c_t is an automorphism with inverse σ^c_{-t}.

To prove continuity, let $a\in C^*(E)$, $t\in\mathbb{R}$ and $\varepsilon>0$. Take x to be a finite sum $x=\sum_{\mu,\nu\in E^*}\lambda_{\mu,\nu}s_\mu s_\nu^*$ such that $\|a-x\|<\varepsilon/3$. For each pair of paths μ,ν with $\lambda_{\mu,\nu}\neq 0$, there is $\delta_{\mu,\nu}$ such that

$$|c(\mu)^{it}c(\nu)^{-it}-c(\mu)^{iu}c(\nu)^{-iu}|<\frac{\varepsilon}{3\sum_{\mu,\nu\in E^*}\|\lambda_{\mu,\nu}s_\mu s_\nu^*\|}$$

for all $u\in\mathbb{R}$ with $|t-u|<\delta_{\mu,\nu}$. If we take δ to be the minimum of all such $\delta_{\mu,\nu}$, then for all $u\in\mathbb{R}$ with $|t-u|<\delta$ we have

$$\|\sigma^c_t(x)-\sigma^c_u(x)\|=\left\|\sum_{\mu,\nu\in E^*}(c(\mu)^{it}c(\nu)^{-it}-c(\mu)^{iu}c(\nu)^{-iu})\lambda_{\mu,\nu}s_\mu s_\nu^*\right\|<$$
$$\frac{\varepsilon}{3\sum_{\mu,\nu\in E^*}\|\lambda_{\mu,\nu}s_\mu s_\nu^*\|}\sum_{\mu,\nu\in E^*}\|\lambda_{\mu,\nu}s_\mu s_\nu^*\|=\frac{\varepsilon}{3}$$

and hence

$$\|\sigma^c_t(a)-\sigma^c_u(a)\|=\|\sigma^c_t(a)-\sigma^c_t(x)+\sigma^c_t(x)-\sigma^c_u(x)+\sigma^c_u(x)-\sigma^c_u(a)\|\leq$$
$$\leq\|\sigma^c_t(a-x)\|+\|\sigma^c_t(x)-\sigma^c_u(x)\|+\|\sigma^c_u(x-a)\|\leq\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.$$

□

From now on, we will write simply σ instead of σ_c. The next result shows that KMS states on $C^*(E)$ are determined by their values at the core algebra.

Proposition 3.2. *Suppose $c:E^1\to\mathbb{R}^*_+$ is such that $c(\mu)\neq 1$ for all $\mu\in E^*\backslash E^0$. If two (σ,β)-KMS states φ_1,φ_2 on $C^*(E)$ coincide at the core algebra $C^*(E)^\gamma$, then $\varphi_1=\varphi_2$.*

Proof. Taking an arbitrary $s_\mu s_\nu^*$ such that $s(\mu)=s(\nu)$, if $|\mu|=|\nu|$ then $s_\mu s_\nu^*\in C^*(E)^\gamma$ and thus $\varphi_1(s_\mu s_\nu^*)=\varphi_2(s_\mu s_\nu^*)$.

Suppose then that $|\mu|\neq|\nu|$, and denote the functional $\varphi_2-\varphi_1$ by φ. Using the KMS condition, one obtains

$$\varphi(s_\mu s_\nu^*)=\varphi(s_\nu^*c(\mu)^{-\beta}s_\mu)=\begin{cases} c(\mu)^{-\beta}\varphi(s^*_{\nu'}) & \text{if } \nu=\mu\nu' \\ c(\mu)^{-\beta}\varphi(s^*_{\mu'}) & \text{if } \mu=\nu\mu' \\ 0 & \text{otherwise}\end{cases}.$$

It is therefore sufficient to show that $\varphi(s_\mu)=\varphi(s_\mu^*)=0$ if $|\mu|\geq 1$. To see this, notice that if $C^*(E)$ has a unit, then

$$\varphi(s_\mu)=\varphi(s_\mu 1)=\varphi(1c(\mu)^{-\beta}s_\mu)=c(\mu)^{-\beta}\varphi(s_\mu),$$

whence $\varphi(s_\mu)=0$ since $c(\mu)\neq 1$ by hypothesis; the non-unital case is established analogously with the use of an approximate unit. □

Theorem 3.3. *Suppose $c:E^1\to\mathbb{R}^*_+$ is such that $c(\mu)\neq 1$ for all $\mu\in E^*\backslash E^0$. If φ is a (σ,β)-KMS state on $C^*(E)$ then its restriction $\omega=\varphi|_{C^*(E)^\gamma}$ to $C^*(E)^\gamma$ satisfies*

$$\omega(s_\mu s_\nu^*)=[\mu=\nu]c(\mu)^{-\beta}\omega(p_{s(\mu)});\tag{3.1}$$

conversely, if ω is a state on $C^*(E)^\gamma$ satisfying (3.1) *then $\varphi = \omega\circ\Phi$ is a (σ,β)-KMS state on $C^*(E)$, where Φ is the conditional expectation from proposition 2.9. The correspondence thus obtained is bijective and preserves convex combinations.*

Proof. Let φ be a (σ,β)-KMS state on $C^*(E)$ and ω its restriction to $C^*(E)^\gamma$. If μ,ν are paths such that $|\mu| = |\nu|$ and $s(\mu) = s(\nu)$ then

$$\omega(s_\mu s_\nu^*) = \varphi(s_\mu s_\nu^*) = \varphi(s_\nu^* \sigma_{i\beta}(s_\mu)) = \varphi(s_\nu^* c(\mu)^{-\beta} s_\mu) =$$
$$= [\mu = \nu] c(\mu)^{-\beta} \varphi(p_{s(\mu)}) = [\mu = \nu] c(\mu)^{-\beta} \omega(p_{s(\mu)}).$$

Conversely, let ω be a state on $C^*(E)^\gamma$ satisfying (3.1) and $\varphi = \omega\circ\Phi$; we have to show that φ satisfies the KMS condition. By continuity and linearity, it is sufficient to verify this for elements $x = s_\mu s_\nu^*$ and $y = s_\zeta s_\eta^*$ where $\mu,\nu,\zeta,\eta \in E^*$ are paths such that $s(\mu) = s(\nu)$ and $s(\zeta) = s(\eta)$.

We need to check that $\varphi(xy) = \varphi(y\sigma_{i\beta}(x))$. First note that

$$xy = (s_\mu s_\nu^*)(s_\zeta s_\eta^*) = \begin{cases} s_{\mu\zeta'} s_\eta^* & \text{if } \zeta = \nu\zeta' \quad (1) \\ s_\mu s_{\eta\nu'}^* & \text{if } \nu = \zeta\nu' \quad (2) \\ 0 & \text{otherwise} \quad (3) \end{cases}$$

and

$$y\sigma_{i\beta}(x) = c(\mu)^{-\beta} c(\nu)^{\beta} (s_\zeta s_\eta^*)(s_\mu s_\nu^*) = c(\mu)^{-\beta} c(\nu)^{\beta} \begin{cases} s_{\zeta\mu'} s_\nu^* & \text{if } \mu = \eta\mu' \quad (a) \\ s_\zeta s_{\nu\eta'}^* & \text{if } \eta = \mu\eta' \quad (b) \\ 0 & \text{otherwise} \quad (c) \end{cases}.$$

There are nine cases to consider. In each case it must be checked whether the resulting paths have the same size, for they will be otherwise sent to 0 by Φ.

Case 1-a. In this case $\zeta = \nu\zeta'$ and $\mu = \eta\mu'$ so that $|\zeta| = |\nu| + |\zeta'|$ and $|\mu| = |\eta| + |\mu'|$. We claim that $|\mu\zeta'| = |\mu| + |\zeta'| = |\eta|$ if and only if $|\zeta\mu'| = |\zeta| + |\mu'| = |\nu|$, and in this case $\mu = \eta$ and $\nu = \zeta$. In fact,

$$|\mu| + |\zeta'| = |\eta| \Leftrightarrow |\eta| + |\mu'| + |\zeta'| = |\eta| \Leftrightarrow |\mu'| + |\zeta'| = 0 \Leftrightarrow$$
$$\Leftrightarrow |\nu| + |\zeta'| + |\mu'| = |\nu| \Leftrightarrow |\zeta| + |\mu'| = |\nu|.$$

Observe that, in this case, we have $|\mu'| + |\zeta'| = 0$ so that $|\mu'| = |\zeta'| = 0$, and hence $\mu = \eta$, $\nu = \zeta$.

It follows that, if $|\mu\zeta'| \neq |\eta|$, then

$$\varphi(xy) = \omega\circ\Phi(xy) = \omega(0) = \omega\circ\Phi(y\sigma_{i\beta}(x)) = \varphi(y\sigma_{i\beta}(x))$$

and, if $|\mu\zeta'| = |\eta|$, we get

$$\varphi(xy) = \varphi(s_\mu s_\mu^*) = \omega(s_\mu s_\mu^*) = c(\mu)^{-\beta}\omega(p_{s(\mu)})$$

and on the other hand

$$\varphi(y\sigma_{i\beta}(x)) = c(\mu)^{-\beta} c(\nu)^{\beta} \varphi(s_\nu s_\nu^*) = c(\mu)^{-\beta} c(\nu)^{\beta} \omega(s_\nu s_\nu^*) =$$
$$= c(\mu)^{-\beta} c(\nu)^{\beta} c(\nu)^{-\beta} \omega(p_{s(\mu)}) = c(\mu)^{-\beta}\omega(p_{s(\mu)}).$$

Case 1-b. Now, we have that $\zeta = \nu\zeta'$ and $\eta = \mu\eta'$ so that $|\zeta| = |\nu\zeta'| = |\nu| + |\zeta'|$ and $|\eta| = |\mu\eta'| = |\mu| + |\eta'|$; as before, we can check that $|\mu| + |\zeta'| = |\eta|$ if and only if $|\zeta| = |\nu| + |\eta'|$. If that is not the case then $\varphi(xy) = 0 = \varphi(y\sigma_{i\beta}(x))$. If the equivalent conditions are true then

$$\varphi(xy) = \varphi(s_{\mu\zeta'} s_\eta^*) = \omega(s_{\mu\zeta'} s_\eta^*) = [\mu\zeta' = \eta] c(\eta)^{-\beta}\omega(p_{s(\eta)})$$

and

$$\varphi(y\sigma_{i\beta}(x)) = c(\mu)^{-\beta}c(\nu)^{\beta}\varphi(s_\zeta s_{\nu\eta'}^*) = c(\mu)^{-\beta}c(\nu)^{\beta}[\zeta = \nu\eta']c(\zeta)^{-\beta}\omega(p_{s(\zeta)}).$$

Since $\zeta = \nu\zeta'$ and $\eta = \mu\eta'$, we have that $\mu\zeta' = \eta$ if and only if $\zeta = \nu\eta'$ and if both are true, then $\zeta' = \eta'$ and

$$c(\mu)^{\beta}c(\nu)^{-\beta}c(\zeta)^{-\beta} = c(\mu)^{-\beta}c(\nu)^{\beta}c(\nu)^{-\beta}c(\eta')^{-\beta} = c(\mu)^{-\beta}c(\eta')^{-\beta} =$$

$$= c(\mu)^{-\beta}c(\zeta')^{-\beta} = c(\eta)^{-\beta}.$$

From our original hypothesis, we have that $s(\eta) = s(\zeta)$ so we conclude that $\varphi(xy) = \varphi(y\sigma_{i\beta}(x))$.

Case 1-c. In this case $\varphi(y\sigma_{i\beta}(x)) = 0$, so we need to check that $\varphi(xy) = 0$. As with the previous case, we have that $\varphi(xy) = [\mu\zeta' = \eta]c(\eta)^{-\beta}\omega(p_{s(\eta)})$; however, in case (c) $\mu\zeta' \neq \eta$ for all ζ' and therefore $\varphi(xy) = 0$.

The other cases are analogous to these three, except for case 3-c, where $\varphi(xy) = 0 = \varphi(y\sigma_{i\beta}(x))$ since $xy = 0 = y\sigma_{i\beta}(x)$.

That the correspondence obtained is bijective follows from Proposition 3.2 and that it preserves convex combinations is immediate. □

Next, we want to show that there is also a bijective correspondence between (σ,β)-KMS states on $C^*(E)$ and a certain class of tracial states on $C_0(E^0)$. We build this correspondence by first describing a correspondence between this class of tracial states on $C_0(E^0)$ and states ω on $C^*(E)^\gamma$ satisfying (3.1).

The conditions found for the states on $C_0(E^0)$ are similar to those in [**11**], although as discussed in [**9**], their results cannot be used directly for an arbitrary graph; nevertheless, the results of Theorem 1.1 of [**11**] still apply in the general setting, and we use them to build a certain kind of transfer operator on the dual of $C_0(E^0)$.

Let us first recall how to construct $C^*(E)$ as C*-algebra associated to a C*-correspondence [**10**]. If we let $A = C_0(E^0)$, then $C_c(E^1)$ has a pre-Hilbert A-module structure given by

$$\langle \xi, \eta\rangle (v) = \sum_{e\in s^{-1}(v)} \overline{\xi(e)}\eta(e) \quad \text{for} \quad v \in E^0,$$

$$(\xi a)(e) = \xi(e)a(s(e)) \quad \text{for} \quad e \in E^1,$$

where $\xi, \eta \in C_c(E^1)$ and $a \in A$; it follows that the completion X of $C_c(E^1)$ with respect to the norm given by $\|\xi\| = \| \langle \xi, \xi\rangle \|^{1/2}$ is a Hilbert A-module. A representation $i_X : A \to \mathcal{L}(X)$ is then defined by by

$$i_X(a)(\xi)(e) = a(r(e))\xi(e) \quad \text{for} \quad v \in E^0,$$

where $\mathcal{L}(X)$ is the C*-algebra of adjointable operators on X.

Let $\mathcal{K}(X)$ be the C*-subalgebra of $\mathcal{L}(X)$ generated by the operators $\theta_{\xi,\eta}$ given by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta\rangle$. For each $e \in E^1$, let $\chi_e \in C_c(E^1)$ be the characteristic function of $\{e\}$ and observe that

$$\left\{ t_\lambda = \sum_{e\in\lambda} \theta_{\chi_e,\chi_e} \right\}_{\lambda\in\Lambda},$$

where Λ is the set of all finite subsets of E^1, is an approximate unit of $\mathcal{K}(X)$. It is essentially the same approximate unit given by Lemma 2.10.

If τ is a tracial state on $C_0(E^0)$, as in Theorem 1.1 of [**11**] we define a trace Tr_τ on $\mathcal{L}(X)$ by

$$\mathrm{Tr}_\tau(T) = \lim_{\lambda\to\infty} \sum_{e\in\lambda} \tau\left(\langle \chi_e, T\chi_e\rangle\right)$$

where $T \in \mathcal{L}(X)$.

For a function $c : E^1 \to \mathbb{R}^*_+$ as in the beginning of the section and $\beta > 0$, we have that $c^{-\beta} \in C_b(E^1)$ and so it defines an operator on $\mathcal{L}(X)$ by pointwise multiplication.

Definition 3.4. Given c and β as above and τ a tracial state on $C_0(E^0)$, we define a trace $\mathcal{F}_{c,\beta}(\tau)$ on $C_0(E^0)$ by

$$\mathcal{F}_{c,\beta}(\tau)(a) = \mathrm{Tr}_\tau(i_X(a)c^{-\beta}).$$

Now, observe that $C_0(E^0) \cong \overline{\mathrm{span}}\{p_v\}_{v\in E^0}$; regarding this as an equality, for a given tracial state τ on $C_0(E^0)$ we will write $\tau(p_v) = \tau_v$. For $v \in E^0$, it can be verified that

$$\mathcal{F}_{c,\beta}(\tau)(p_v) = \lim_{D\to r^{-1}(v)} \sum_{e\in D} c(e)^{-\beta}\tau_{s(e)},$$

where the limit is taken on finite subsets D of $r^{-1}(v)$, and $\mathcal{F}_{c,\beta}(\tau)(p_v) = 0$ if $r^{-1}(v) = \emptyset$.

Remark 3.5. By Theorem 1.1 of [**11**], if $\mathcal{F}_{c,\beta}(\tau)(a) < \infty$ for all $a \in C_0(E^0)$, then $\mathcal{F}_{c,\beta}(\tau)$ is actually a positive linear functional; also, if $V \subseteq E^0$ and $\mathcal{F}_{c,\beta}(\tau)(p_v) < \infty$ for all $v \in V$ then $\mathcal{F}_{c,\beta}(\tau)$ is a positive linear functional on $\overline{\mathrm{span}}\{p_v : v \in V\}$.

Definition 3.6. For a vertex $v \in E^0$ and a positive integer n, we define

$$r^{-n}(v) = \{\mu \in E^n : r(\mu) = v\}.$$

Lemma 3.7. *If $\mathcal{F}_{c,\beta}(\tau)(p_v) \le \tau_v$ for all $v \in E^0$ then*

$$\lim_{D\to r^{-n}(v)} \sum_{\mu\in D} c(\mu)^{-\beta}\tau_{s(\mu)} \le \tau_v$$

for all $v \in E^0$ and for all $n \in \mathbb{N}^*$.

Proof. This is proved by induction. The case $n = 1$ is the hypothesis. Now suppose it is true for n, then

$$\lim_{D\to r^{-(n+1)}(v)} \sum_{\mu\in D} c(\mu)^{-\beta}\tau_{s(\mu)} = \lim_{D\to r^{-n}(v)} \sum_{\nu\in r^{\le n}} c(\nu)^{-\beta} \sum_{e\in r^{-1}(s(\nu))} c(e)^{-\beta}\tau_{s(e)}[\nu e \in D] \le$$

$$\le \lim_{D\to r^{-n}(v)} \sum_{\nu\in D} c(\nu)^{-\beta}\tau_{s(\nu)} \le \tau_v$$

where the first inequality is true due to the fact that since c is a positive function then the net $\sum_{e\in D} c(e)^{-\beta}\tau_{s(e)}$ for finite subsets D of $r^{-1}(s(\nu))$ is nondecreasing and less than or equal to $\tau_{s(\nu)}$ by hypothesis. The last inequality is the induction hypothesis. ◻

The next lemma is found in [**7**] for unital algebras, but their proof carries out the same in the non-unital case by using an approximate unit instead of a unit.

Lemma 3.8 (Exel-Laca). *Let B be a C^*-algebra, A be a C^*-subalgebra such that an approximate unit of A is also an approximate unit of B and I a closed bilateral ideal of B such that $B = A + I$. Let φ be a state on A and ψ a linear positive functional on I such that $\varphi(x) = \psi(x)$ $\forall x \in A \cap I$ and $\overline{\psi}(x) \leq \varphi(x)$ $\forall x \in A^+$, where $\overline{\psi}(x) = \lim_\lambda \psi(bu_\lambda)$ for an approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ of I. Then there is a unique state Φ on B such that $\Phi|_A = \varphi$ and $\Phi|_I = \psi$.*

We want to use this lemma for $A = C_n$, $I = F_{n+1}$ and $B = C_{n+1}$, defined in section 2. For that, we first note that F_{n+1} is indeed an ideal of C_{n+1} by Proposition 2.11 and that the approximate unit for F_0 given by Lemma 2.10 is also an approximate unit of C_n for all n. We also need to know what the intersection $A \cap I$ is, and for that we need a preliminary result.

Lemma 3.9. *Suppose c, β and τ are such that $\mathcal{F}_{c,\beta}(\tau)(a) \leq \tau(a)$ for all $a \in C_0(E^0)^+$, then for each $k \geq 1$ there is a unique positive linear functional ψ_k on F_k defined by*

$$\psi_k(s_\mu s_\nu^*) = [\mu = \nu]c(\mu)^{-\beta}\tau_{s(\mu)}. \tag{3.2}$$

Proof. Since $\{s_\mu s_\nu^* : \mu,\nu \in E^k,\ s(\mu) = s(\nu)\}$ is linearly independent, equation 3.2 defines a unique linear functional on $\operatorname{span}\{s_\mu s_\nu^* : \mu,\nu \in E^k,\ s(\mu) = s(\nu)\}$. To extend to the closure, it is sufficient to prove that ψ_k is continuous.

If $x \in \operatorname{span}\{s_\mu s_\nu^* : \mu,\nu \in E^k,\ s(\mu) = s(\nu)\}$ then

$$x = \sum_{v\in V}\sum_{(\mu,\nu)\in G_v} a^v_{\mu,\nu} s_\mu s_\nu^*$$

where V is a finite subset of E^0 and G_v is a finite subset of $\{(\mu,\nu) \in E^n \times E^n : s(\mu) = s(\nu) = v\}$. Using the decomposition given by equation 2.1 and observing that $\{s_\mu s_\nu^* : (\mu,\nu) \in G_v\}$ can be completed to generators of a matrix algebra, we have that

$$\|x\| = \max_{v\in V}\left\|\sum_{(\mu,\nu)\in G_v} a^v_{\mu,\nu} s_\mu s_\nu^*\right\| = \max_{v\in V}\|(a^v_{\mu,\nu})_{\mu,\nu}\|$$

where the last norm is the matrix norm.

If Tr is the usual matrix trace we have

$$|\psi_k(x)| = \left|\psi_k\left(\sum_{v\in V}\sum_{(\mu,\nu)\in G_v} a^v_{\mu,\nu} s_\mu s_\nu^*\right)\right| =$$

$$= \left|\sum_{v\in V}\sum_{(\mu,\nu)\in G_v} a^v_{\mu,\nu}[\mu=\nu]c(\mu)^{-\beta}\tau_{s(\mu)}\right| =$$

$$= \left|\sum_{v\in V}\operatorname{Tr}((a^v_{\mu,\nu})_{\mu,\nu}\operatorname{diag}(c(\mu)^{-\beta}\tau_{s(\mu)})\right| \leq$$

$$\leq \sum_{v\in V}\left|\operatorname{Tr}((a^v_{\mu,\nu})_{\mu,\nu}\operatorname{diag}(c(\mu)^{-\beta}\tau_{s(\mu)})\right| \leq$$

$$\leq \sum_{v\in V}\|(a^v_{\mu,\nu})_{\mu,\nu}\|\sum_{\mu:(\mu,\mu)\in G_v} c(\mu)^{-\beta}\tau_{s(\mu)} \leq^{\text{lemma } 3.7}$$

$$\leq \sum_{v\in V} \|(a^v_{\mu,\nu})_{\mu,\nu}\|\tau_v \leq \max_{v\in V}(\|(a^v_{\mu,\nu})_{\mu,\nu}\|) \sum_{v\in V} \tau_v =$$

$$= \|x\| \sum_{v\in V} \tau_v \leq \|x\|$$

where the last inequality comes from the fact that τ comes from a probability measure on a discrete space. $\square$

THEOREM 3.10. *If ω is a state on $C^*(E)^\gamma$ satisfying (3.1) then its restriction τ to $C_0(E^0)$ satisfies:*

(K1) *$\mathcal{F}_{c,\beta}(\tau)(a) = \tau(a)$ for all $a \in \overline{\text{span}}\{p_v : 0 < |r^{-1}(v)| < \infty\}$,*
(K2) *$\mathcal{F}_{c,\beta}(\tau)(a) \leq \tau(a)$ for all $a \in C_0(E^0)^+$.*

Conversely, if τ is a tracial state on $C_0(E^0)$ satisfying (K1) and (K2) then there is unique state ω on $C^*(E)^\gamma$ satisfying (3.1). This correspondence preserves convex combinations.

PROOF. Let ω be a state on $C^*(E)^\gamma$ satisfying (3.1) and τ its restriction to $C_0(E^0)$. By Remark 3.5, to establish (K1) it is sufficient to consider $a = p_v$ where $v \in E^0$ is such that $0 < |r^{-1}(v)| < \infty$, and in this case

$$\tau(p_v) = \omega(p_v) = \omega\left(\sum_{e\in r^{-1}(v)} s_e s_e^*\right) = \sum_{e\in r^{-1}(v)} c(e)^{-\beta}\omega(p_{s(e)}) =$$

$$= \sum_{e\in r^{-1}(v)} c(e)^{-\beta}\tau_{s(e)} = \mathcal{F}_{c,\beta}(\tau)(p_v).$$

For (K2), let $a \in C_0(E^0)^+$ and write $a = \sum_{v\in E^0} a_v p_v$; again, by remark 3.5 it is sufficient to show the result for $a = p_v$ where $v \in E^0$. If $0 < |r^{-1}(v)| < \infty$, then we have an equality as shown above. If $|r^{-1}(v)| = 0$, then $\mathcal{F}_{c,\beta}(\tau)(p_v) = 0 \leq \tau(p_v)$. If $|r^{-1}(v)| = \infty$, then

$$\mathcal{F}_{c,\beta}(\tau)(p_v) = \lim_{D\to r^{-1}(v)} \sum_{e\in D} c(e)^{-\beta}\tau_{s(e)} = \lim_{D\to r^{-1}(v)} \sum_{e\in D} \omega(s_e s_e^*) =$$

$$= \lim_{D\to r^{-1}(v)} \sum_{e\in D} \omega(p_v s_e s_e^*) \leq \omega(p_v) = \tau(p_v).$$

To see the inequality above, we observe that $s_e s_e^*$ are mutually orthogonal projections that commute with p_v so that

$$p_v - \sum_{e\in D} p_v s_e s_e^* = p_v\left(1 - \sum_{e\in D} s_e s_e^*\right) = \left(1 - \sum_{e\in D} s_e s_e^*\right) p_v \left(1 - \sum_{e\in D} s_e s_e^*\right) \geq 0.$$

Now, let τ be a tracial state on $C_0(E^0)$ satisfying (K1) and (K2). We will use Lemma 3.8 and the discussion after it. Observe that $F_0 = C_0(E^0)$ and let $\psi_0 = \tau$. For $n \geq 1$, by Lemma 3.9 there exists a positive linear functional ψ_n on F_n defined by

$$\psi_n(s_\mu s_\nu^*) = [\mu = \nu]c(\mu)^{-\beta}\tau_{s(\mu)}.$$

Let us show by induction that there is a unique state φ_n on C_n such that the restriction to F_n is ψ_n. For $n = 1$, we use Lemma 3.8 with $A = C_0(E^0)$, $I = F_1$,

$B = C_1$, $\varphi = \tau$ and $\psi = \psi_1$. By Proposition 2.12, in this case $A \cap I = \overline{\operatorname{span}}\{p_v : v \in E^0,\ 0 < |r^{-1}(v)| < \infty\}$ and if $p_v \in A \cap I$ then

$$\psi(p_v) = \psi_1(p_v) = \psi_1(s_v s_v^*) = \tau_v = \tau(p_v).$$

Using the approximate unit given by Lemma 2.10, for any $v \in E^0$ we have

$$\overline{\psi}(p_v) = \overline{\psi_1}(p_v) = \lim_{\lambda\to\infty} \psi_1(p_v u_\lambda) = \lim_{D\to r^{-1}(v)} \sum_{e\in D} \psi_1(s_e s_e^*) =$$

$$= \lim_{D\to r^{-1}(v)} \sum_{e\in D} c(e)^{-\beta}\tau_{s(e)} = \mathcal{F}_{c,\beta}(\tau)(p_v) \le \tau(p_v),$$

where the last inequality is exactly (K2).

Now suppose that there is a unique state φ_n on C_n such that the restriction to F_n is ψ_n and let us show that this is also true for $n+1$. We set $A = C_n$, $I = F_{n+1}$, $B = C_{n+1}$, $\varphi = \varphi_n$ and $\psi = \psi_{n+1}$ on Lemma 3.8. By Proposition 2.12, we have that $A \cap I = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^n, s(\mu) = s(\nu), |\mu| = |\nu|, 0 < |r^{-1}(s(\mu))| < \infty\}$. Let $s_\mu s_\nu^* \in A \cap I$. Since $0 < |r^{-1}(s(\mu))| < \infty$ we have that

$$\psi(s_\mu s_\nu^*) = \psi_{n+1}(s_\mu s_\nu^*) = \sum_{e\in r^{-1}(s(\mu))} \psi_{n+1}(s_{\mu e} s_{\nu e}^*) = \sum_{e\in r^{-1}(s(\mu))} [\mu e = \nu e] c(\mu e)^{-\beta}\tau_{s(\mu e)} =$$

$$= \sum_{e\in r^{-1}(s(\mu))} [\mu = \nu] c(\mu)^{-\beta} c(e)^{-\beta}\tau_{s(e)} = [\mu = \nu] c(\mu)^{-\beta} \sum_{e\in r^{-1}(s(\mu))} c(e)^{-\beta}\tau_{s(e)} =$$

$$= [\mu = \nu] c(\mu)^{-\beta}\mathcal{F}_{c,\beta}(\tau)(p_{s(\mu)}) = [\mu = \nu] c(\mu)^{-\beta}\tau(p_{s(\mu)}) = \psi_n(s_\mu s_\nu^*) = \varphi_n(s_\mu s_\nu^*).$$

Again, using the approximate unit given by Lemma 2.10, if $s_\mu s_\nu^* \in C_n$, then

$$\overline{\psi}(s_\mu s_\nu^*) = \overline{\psi_{n+1}}(s_\mu s_\nu^*) = \lim_{\lambda\to\infty} \psi_{n+1}(s_\mu s_\nu^* u_\lambda) = \lim_{D\to r^{\le n+1-|\nu|}(s(\mu))} \sum_{\zeta\in D} \psi_{n+1}(s_{\mu\zeta} s_{\nu\zeta}^*) =$$

$$= \lim_{D\to r^{\le n+1-|\nu|}(s(\nu))} \sum_{\zeta\in D} [\mu\zeta = \nu\zeta] c(\nu\zeta)^{-\beta}\tau_{s(\nu\zeta)} =$$

$$= \lim_{D\to r^{\le n+1-|\nu|}(s(\nu))} \sum_{\zeta\in D} [\mu = \nu] c(\nu)^{-\beta} c(\zeta)^{-\beta}\tau_{s(\zeta)} =$$

$$= [\mu = \nu] c(\nu)^{-\beta} \lim_{D\to r^{\le n+1-|\nu|}(s(\nu))} \sum_{\zeta\in D} c(\zeta)^{-\beta}\tau_{s(\zeta)} \le$$

$$\le [\mu = \nu] c(\nu)^{-\beta}\tau_{s(\nu)} = \varphi_n(s_\mu s_\nu^*),$$

where the inequality is given by Lemma 3.7, which is a consequence of (K2).

By the description of the core $C^*(E)^\gamma$ as an inductive limit of the C_n, we can define a state ω as the inductive limit of φ_n. By construction, ω satisfies (3.1) and, since each φ_n is uniquely defined by (3.1), so is ω.

Finally, it is easily seen that the correspondence built preserves convex combinations by construction.

□

4. Ground states

In this section, we let a function $c : E^1 \to \mathbb{R}_+^*$ be given and define a one-parameter group of automorphisms σ as in the last section.

The following definition of a ground state will be used [**3**].

Definition 4.1. We say that ϕ is a σ-*ground state* if for all $a, b \in C^*(E)^a$, the entire analytic function $\zeta \mapsto \phi(a\sigma_\zeta(b))$ is uniformly bounded in the region $\{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) \geq 0\}$, where $C^*(E)^a$ is the set of analytic elements for σ.

Proposition 4.2. *If τ is a tracial state on $C_0(E^0)$ such that* $\mathrm{supp}(\tau) \subseteq \{v \in E^0 : v \text{ is singular}\}$ *then there is a unique state ϕ on $C^*(E)$ such that*

(i) $\phi(p_v) = \tau(p_v)$ *for all* $v \in E^0$*;*
(ii) $\phi(s_\mu s_\nu^*) = 0$ *if* $|\mu| > 0$ *or* $|\nu| > 0$.

Proof. First, observe that a state ϕ satisfying (ii) is uniquely determined by its values on $C^*(E)^\gamma$ because (ii) implies that $\phi = \phi|_{C^*(E)^\gamma} \circ \Phi$, where Φ is the conditional expectation given by Proposition 2.9.

Given τ as in the statement of the proposition, a state ω on $C^*(E)^\gamma$ can be built in the same way as in the proof of Theorem 3.10. For each n, use Lemma 3.8 with $A = C_n$, $B = C_{n+1}$, $I = F_{n+1}$, $\psi_n \equiv 0$ and φ_n is given by the previous step, where for the first step we have $\varphi_0 = \tau$. For $\omega = \varinjlim \varphi_n$, we have that $\phi = \omega \circ \Phi$ satisfies (i) and (ii) and is unique by construction. □

Proposition 4.3. *If c is such that $c(e) > 1$ for all $e \in E^1$, then a state ϕ on $C^*(E)$ is a σ-ground state for σ if and only if $\phi(s_\mu s_\nu^*) = 0$ whenever $|\mu| > 0$ or $|\nu| > 0$.*

Proof. If ϕ is a ground state then for each pair $\mu, \nu \in E^*$ the function $\zeta \mapsto |\phi(s_\mu \sigma_\zeta(s_\nu^*))|$ is bounded on the upper half of the complex plane. If $\zeta = x + iy$ then

$$|\phi(s_\mu \sigma_\zeta(s_\nu^*))| = |\phi(s_\mu c(\nu)^{-i\zeta} s_\nu^*)| = |c(\nu)^{y-ix} \phi(s_\mu s_\nu^*)| = c(\nu)^y |\phi(s_\mu s_\nu^*)|.$$

If $|\nu| > 0$, we have that $c(\nu) > 1$ and so the only possibility for the above function to be bounded is if $\phi(s_\mu s_\nu^*) = 0$. It is shown analogously that if $|\mu| > 0$ then $\phi(s_\mu s_\nu^*) = 0$.

For the converse, observe that if $|\mu| = |\nu| = 0$ then $|\phi(s_\mu \sigma_\zeta(s_\nu^*))| = |\phi(s_\mu s_\nu^*| \leq 1$. It can be now readily verified that if $\phi(s_\mu s_\nu^*) = 0$ whenever $|\mu| > 0$ or $|\nu| > 0$ then ϕ is a ground state. □

Theorem 4.4. *If c is such that $c(e) > 1$ for all $e \in E^1$ then there is a bijective correspondence, given by restriction, between σ-ground states ϕ and tracial states τ on $C_0(E^0)$ such that* $\mathrm{supp}(\tau) \subseteq \{v \in E^0 : v \text{ is singular}\}$.

Proof. This is an immediate consequence of Propositions 4.2 and 4.3. Just note that if ϕ is a σ-ground state and $v \in E^0$ is not singular then

$$\phi(p_v) = \phi\left(\sum_{e \in r^{-1}(v)} s_e s_e^*\right) = 0.$$

□

5. Examples

In this section we give two examples with infinite graphs and study the KMS states on the C*-algebras associated to these graphs.

Example 5.1 (The Cuntz algebra $\mathcal{O}_\infty$). Let $E^0 = \{v\}$ be any unitary set and $E^1 = \{e_n\}_{n\in\mathbb{N}}$ any countably infinite set with $r(e_n) = s(e_n) = v$ $\forall n \in \mathbb{N}$, then $C^*(E) \cong \mathcal{O}_\infty$.

If $c(e_n) = e$ (Euler's number) then we have the usual gauge action. In this case, $\mathcal{F}_{c,\beta}(\tau)(p_v) = \infty$ so that condition (K2) from Theorem 3.10 is not satisfied and we have no KMS states for finite β. Since we have only one state on $C_0(E^0)$ and v is a singular vertex, by Theorem 4.4 there exists a unique ground state.

Now if $c(e_n) = a_n$ where $a_n \in (1,\infty)$ is such that there is $\beta > 0$ for which $\sum_{n=0}^\infty a_n^{-\beta}$ converges, then there exists $\beta_0 > 0$ such that $\sum_{n=0}^\infty a_n^{-\beta} = 1$. Observing that $\mathcal{F}_{c,\beta}(\tau)(p_v) = \sum_{n=0}^\infty a_n^{-\beta}$ and using again the fact that there exists only one state on $C_0(E^0)$, we conclude from Theorems 3.3 and 3.10 that there is no KMS state for $\beta < \beta_0$, there exists a unique KMS state for each $\beta \geq \beta_0$ and, as with the gauge action, there is a unique ground state.

Example 5.2 (A graph with infinitely many sources). Let $E^0 = \{v_n\}_{n\in\mathbb{N}}$ and $E^1 = \{e_n\}_{n\in\mathbb{N}\setminus\{0\}}$ be countably infinite sets and define $r(e_n) = v_0$ and $s(e_n) = v_n$ for all $n \in \mathbb{N}\setminus\{0\}$.

Again, let $a_n \in (1,\infty)$, $n \in \mathbb{N}\setminus\{0\}$, be such that $\sum_{n=1}^\infty a_n^{-\beta}$ converges for some $\beta > 0$. For $n \neq 0$ we have that $\mathcal{F}_{c,\beta}(\tau)(p_{v_n}) = 0$ and for $n = 0$ we have $\mathcal{F}_{c,\beta}(\tau)(p_{v_0}) = \sum_{n=1}^\infty a_n^{-\beta}\tau_{v_n}$. Condition (K1) of Theorem 3.10 is trivially satisfied, and for condition (K2) we need $\sum_{n=1}^\infty a_n^{-\beta}\tau_{v_n} \leq \tau_{v_0}$.

If $\tau_{v_0} > 0$, since $0 \leq \tau_{v_n} \leq 1$ for all n there exists $\beta_0 > 0$ such that $\sum_{n=1}^\infty a_n^{-\beta_0}\tau_{v_n} = \tau_{v_0}$ so that (K2) is verified for all $\beta \geq \beta_0$ and so there are infinitely many KMS states. And for $\beta < \beta_0$ (K2) is not verified so that there are no KMS states.

For ground states, since all vertices are singular, we have no restriction on τ_{v_0}; every state τ on $C_0(E^0)$ gives a ground state on $C^*(E)$.

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